Perverse Coherent Sheaves on Blow-up

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BASED ON JOINT WORKS WITH KŌTA YOSHIOKA (KOBE)

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§ t-structure & torsion theory (REVIEW)

$\mathcal{A}$: abelian category $\rightsquigarrow$ $D^b(\mathcal{A})$: derived category

We know many examples

$\boxed{D^b(\mathcal{A}) \text{ is equivalent to } D^b(\mathcal{A}') \ (\mathcal{A}' : \text{different abelian category}).}$

<u>Examples</u>

- Fourier-Mukai functor $\quad D^b(\mathrm{Coh}\,A) \cong D^b(\mathrm{Coh}\,A')$
  A : abelian variety $\quad A'$: dual
- (categorical) McKay correspondence $\quad D^b(\mathrm{Coh}^{\Gamma}(X)) \cong D^b(\mathrm{Coh}\,M)$
  M : crepant resolution of X/Γ
- tilting sheaf $\quad D^b(\mathrm{Coh}\,X) \cong D^b(\mathrm{mod}\,A)$
  A : (finite dimensional) noncommutative algebra

<u>subexample</u>

$$D^b(\mathrm{Coh}\,\mathbb{P}^1) \cong D^b(\mathrm{mod}(\text{Kronecker quiver}))$$

$$\circ \rightrightarrows \circ$$

Probably the most famous one (among representation theorists)

Riemann-Hilbert correspondence (Kashiwara = 柏原)

$D^b_{rh}(D_X) \cong D^b(\mathrm{Cons}(X))$ X : complex manifold

regular holonomic D-modules constructible sheaves

$\rightsquigarrow$ proof of Kazhdan-Lusztig conjecture
character formulas of ∞-dim'l rep. of $\mathfrak{g}$
(one of deepest results in RT)

In these examples $D^b(\mathcal{A}) \cong D^b(\mathcal{A}')$, we have
an "exotic" abelian category (i.e. $\mathcal{A}'$) inside $D^b(\mathcal{A})$.

- $D^b(\mathrm{Coh}\, A) \supset \mathrm{Coh}\, A'$, $D^b(\mathrm{Coh}\, A') \supset \mathrm{Coh}\, A$
- $D^b(\mathrm{Coh}\, X) \supset \mathrm{mod}\, A$, $D^b(\mathrm{mod}\, A) \supset \mathrm{Coh}\, X$
- $D^b(\mathrm{Cons}\, X) \supset D_X^{rh}$, $D^b_{rh}(D_X) \supset \mathrm{Cons}(X)$

Beilinson-Bernstein-Deligne : axiomized this as t-structures.

<u>Def</u>. $\mathbb{D}$: triangulated category

A t-structure on $\mathbb{D}$ is a pair $(\mathbb{D}^{\leq 0}, \mathbb{D}^{\geq 0})$ of full subcategories such that for $\mathbb{D}^{\leq n} = \mathbb{D}^{\leq 0}[-n]$, $\mathbb{D}^{\geq n} = \mathbb{D}^{\geq 0}[-n]$

1) $\mathbb{D}^{\leq -1} \subset \mathbb{D}^{\leq 0}$, $\mathbb{D}^{\geq 1} \subset \mathbb{D}^{\geq 0}$

2) $X \in \mathbb{D}^{\leq 0}$, $Y \in \mathbb{D}^{\geq 1} \Longrightarrow \mathrm{Hom}_{\mathbb{D}}(X, Y) = 0$

3) $\forall X \in \mathbb{D}$ $\exists$ distinguished triangle

$$\begin{tikzcd} X_0 \arrow[r] & X \arrow[dl] \\ X_1 \arrow[u, "+1"] & \end{tikzcd} \quad \text{s.t.} \quad \begin{array}{l} X_0 \in \mathbb{D}^{\leq 0} \\ X_1 \in \mathbb{D}^{\geq 1} \end{array}$$

(trivial)

<u>Example</u>. $\mathcal{A}$: abelian category $\quad D^b(\mathcal{A})$: derived category

$D^{\geq 0}(\mathcal{A}) := \{ X \in D^b(\mathcal{A}) \mid H^i(X) = 0 \quad \forall i < 0 \}$

$D^{\leq 0}(\mathcal{A}) := \{ \quad " \quad \mid \quad " \quad \forall i > 0 \}$

$\Longrightarrow D^{\geq 0}(\mathcal{A}), D^{\leq 0}(\mathcal{A})$: standard t-structure

But through derived category equivalences, trivial examples give nontrivial examples.

Theorem. (BBD)

$\mathcal{C} := D^{\geq 0} \cap D^{\leq 0}$ heart (or core) of the t-structure is an abelian category.

- For the standard t-structure, the heart $\mathcal{C} = \mathcal{A}$
- $D^b(\mathcal{A}) \cong D^b(\mathcal{A}') \Longrightarrow (D^{\geq 0}(\mathcal{A}'), D^{\leq 0}(\mathcal{A}'))$: exotic t-structure of $D^b(\mathcal{A})$
- In general a t-structure may not come from a derived equivalence.

In $D^b(\mathrm{Cons}\,X) \cong D^b_{rh}(\mathcal{D}_X)$,

the t-structure $(D^{\geq 0}_{rh}(\mathcal{D}_X), D^{\leq 0}_{rh}(\mathcal{D}_X))$ can be defined entirely in the language of $D^b(\mathrm{Cons}\,X)$.

This definition makes sense e.g. when $X/\mathbb{F}_q$: finite field

$\Longrightarrow$ theory of (mixed) perverse sheaves

many applications to RT.
not only the sol. of KL conjecture.

<u>Remark</u>. I do NOT review Bridgeland's stability condition.
But one of several equivariant definitions
is based on t-structures.

stability condition

$=$ t-structure $+$ Z: central charge on $\mathcal{C}$: heart

s.t. Harder-Narasimhan filtration exists

Another example of t-structures (studied in representation theory of noncommutative algebras

<u>Def.</u> $\mathcal{A}$: abelian category

A torsion pair $(\mathcal{T}, \mathcal{F})$ is a pair of full subcategories such that

1) $\mathrm{Hom}(T, F) = 0 \quad \forall T \in \mathcal{T}, \forall F \in \mathcal{F}$

2) $\forall X \in \mathcal{A} \quad \exists$ short exact sequence

$$0 \to \underset{\mathcal{T}}{\overset{\uparrow}{T}} \to X \to \underset{\mathcal{F}}{\overset{\uparrow}{F}} \to 0$$

T = torsion part of X

F = free part of X

<u>Example</u> $\mathcal{A} = \mathrm{Coh}\, X$

$\mathcal{T}$ = torsion sheaves $= \{ E \in \mathrm{Coh}\, X \mid \text{stalk at generic pt} = 0 \}$

$\mathcal{F}$ = torsion free sheaves

$E \in \mathrm{Coh}\, X \Rightarrow \underset{\text{torsion part}}{T(E)} = \{ s \in E \mid fs = 0 \text{ for } \exists f \in \mathcal{O}_X \setminus 0 \}$

Theorem (Happel–Reiten–Smalø)

$\mathbb{D} = D^b(\mathcal{A})$

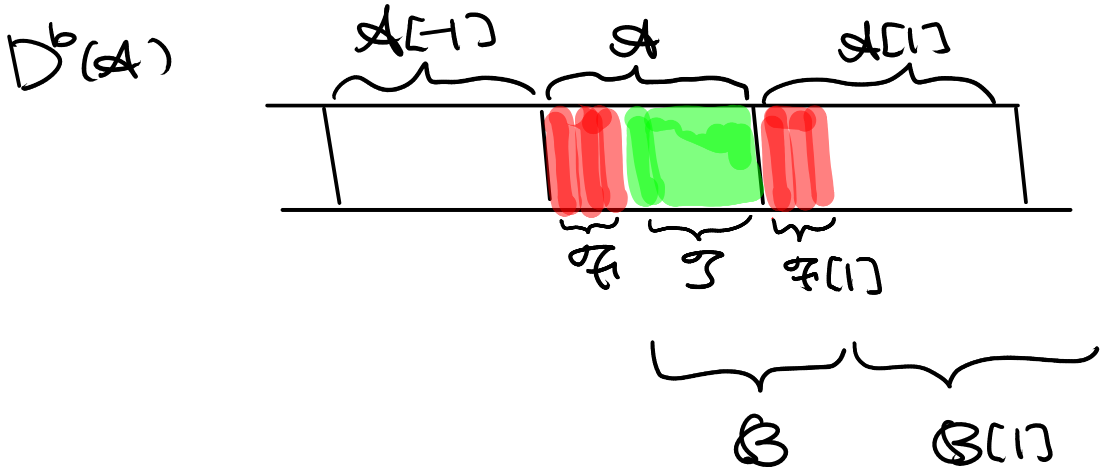
$\mathbb{D}^{\leq 0} := \{ X \in \mathbb{D} \mid H^i(X) = 0 \quad i > 0, \quad H^0(X) \in \mathcal{T} \}$

$\mathbb{D}^{\geq 0} := \{ X \in \mathbb{D} \mid H^i(X) = 0 \quad i < -1, \quad H^{-1}(X) \in \mathcal{F} \}$

$\Longrightarrow \ (\mathbb{D}^{\leq 0}, \mathbb{D}^{\geq 0})$: t-structure

$\mathcal{B} := \mathbb{D}^{\leq 0} \cap \mathbb{D}^{\geq 0}$: abelian category

$(\mathcal{F}[1], \mathcal{T})$: torsion pair in $\mathcal{B}$



Exercise What is $\mathcal{B}$ for $\mathrm{Coh}\,\mathbb{P}^1 \supset$ (torsion, torsion free) ?

Remark. This transition $\mathcal{A} \longrightarrow \mathcal{B}$ occurs when we cross a certain wall in the space of stability conditions.

§ Review of Bridgeland's perverse coherent sheaves
(another source of Bridgeland stability condition)

<u>Side Remark</u>. Several other people use "perverse coherent sheaves" in other context.

<u>Bridgeland's motivation</u>

Want to show $D^b(\mathrm{Coh}\,Y) \overset{\Phi}{\underset{\cong}{\rightarrow}} D^b(\mathrm{Coh}\,Y^+)$ for $Y \searrow X \swarrow Y^+$ flop of 3-folds

via Fourier-Mukai transform

If it is possible, $Y^+ = \{ \Phi^{-1}(\mathcal{O}_y) \mid y \in Y^+ \}$

$\mathcal{O}_y$: skyscraper sheaf

$\therefore$ Y^+ : moduli space of objects in $D^b(\mathrm{Coh}\,Y)$.

To define moduli space, we need to use the geometric invariant theory.
abelian category + slope
(+ quot schemes etc)

He constructed an abelian category $\mathrm{Per}(Y/X) \subset D^b(\mathrm{Coh}\,Y)$

$\mathrm{Per}(Y/X)$: perverse coherent sheaves

$p : Y \to X$ • birational, $\dim p^{-1}(x) \leq 1 \quad \forall x$ (cf. Bryan's talk)
• $\mathbb{R}p_* \mathcal{O}_Y = \mathcal{O}_X$

$\mathcal{C} := \{ K \in \mathrm{Coh}\, Y \mid \mathbb{R}p_* K = 0 \}$
$\mathcal{T} := \{ E \in \mathrm{Coh}\, Y \mid R^1 p_* E = 0,\ \mathrm{Hom}(E, K) = 0 \quad \forall K \in \mathcal{C} \}$
$\mathcal{F} := \{ E \in \mathrm{Coh}\, Y \mid p_* E = 0 \}$

$\Longrightarrow$ $(\mathcal{T}, \mathcal{F})$: torsion pair of $\mathrm{Coh}\, Y$

$$\mathrm{Perv}(Y/X) := \left\{ E \in D^b(\mathrm{Coh}\, Y) \;\middle|\; \begin{array}{l} H^i(E) = 0 \text{ for } i \neq 0, -1 \\ H^0(E) \in \mathcal{T},\ H^{-1}(E) \in \mathcal{F} \end{array} \right\}$$

abelian category

This definition looks artificial, but it turns out to be correct, after some studies ....

Starting observation : $E \in \mathrm{Perv}(Y/X) \Longrightarrow \mathbb{R}p_*(E)$: coherent sheaf on X

In fact, afterwards Van den Bergh : $\mathrm{Perv}(Y/X) \subset \mathrm{Coh}(X)$

$\mathrm{Perv}(Y/X)$ = coherent $\mathcal{A}$-modules, $\mathcal{A}$: noncommutative $\mathcal{O}_X$-algebra

Next we construct moduli space of <u>stable</u> perverse coherent sheaves.

This part is highly technical. A modification of the GIT quotient construction of moduli spaces (of Simpson). Also it will become unnecessary if powerful machinery will be developed in more general situation.

STEP 1. Define a stability condition on $\mathrm{Perv}(Y/X)$ by using Hilbert polynomials.
Under a mild assumption, show that

E : semistable $\Longrightarrow$ $H^{-1}(E) = 0$ i.e., E : coherent sheaf

Very roughly $H^{-1}(E)[1] \subset E$ violates the semistability inequality.

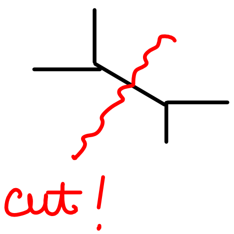
STEP 2. Follow Simpson's argument.

Anyway we have moduli space of stable perverse coherent sheaves $M^P(X/Y)$ and a morphism $M^P(X/Y) \longrightarrow M_H(X)$

Remark.

Perverse coherent sheaves are "semi-global" objects.
They are local w.r.t. X, but not w.r.t. Y.

For example, on the conifold $Y = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \longrightarrow \{xy = zw\} = X$
ordinary ideal sheaves can be studied by the topological vertex



But this is not possible for $\mathrm{Perv}(Y/X)$.

§ Coherent sheaves on blow-up

X : nonsingular projective surface $/\mathbb{C}$

H : ample line bundle ($\leftarrow\cdots\rightarrow$ Kähler metric)

$0 \in X$: a point (fixed)

$$\begin{array}{ccc} p : \hat{X} & \longrightarrow & X \\ \cup & & \cup \\ \mathbb{P}^1 & \longrightarrow & 0 \end{array} \quad \text{blow-up of } X \text{ at } 0$$

locally $\hat{X}$: total space of $\mathcal{O}_{\mathbb{P}^1}(-1)$ over $\mathbb{P}^1$

$$\begin{array}{ccc} \mathrm{Tot}\,\mathcal{O}_{\mathbb{P}^1}(-1) \subset \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2}) = \mathbb{P}^1 \times \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ \cup & & \cup \\ \text{zero section} = \mathbb{P}^1 & \longrightarrow & 0 \end{array}$$

p^*H : not ample since $\langle p^*H, [C] \rangle = 0$

but $H_\varepsilon = p^*H - \varepsilon\, \mathrm{P.D.}[C]$: ample for $\forall \varepsilon > 0$

$\therefore H_\varepsilon$: approximation of p^*H

Problem Study relations between

— moduli space M & H-stable sheaves on X

— " $\hat{M}$ H_ε- " on $\hat{X}$

for sufficiently small $\varepsilon > 0$

Rem. moduli of sheaves ⋯ algebro-geometric model of moduli space of instantons

Thus these are related to Donaldson invariants for C^∞-4-mfd.

(brother of DT invariants)

$\underset{\text{Witten}}{=}$ twisted $N=2$ SUSY Yang-Mills

Why do we bother blow-up ?

- Most basic operation in birational geometry of complex surfaces
  (& smooth 4-manifold)

  more specifically
  Fintushel-Stern's blowup formula of Donaldson invariants
  is given by theta functions $\longleftrightarrow$ Seiberg-Witten curves
  for $N=2$ SUSY YM

  $\rightsquigarrow$ Proof of Nekrasov conjecture on instanton counting
  (N-Yoshioka 2003)
  $\rightsquigarrow$ Proof of the wall-crossing formula of Donaldson invariants
  with cpx surfaces with $p_g = 0$
  via modular forms
  (Göttsche - N - Yoshioka + T. Mochizuki 2006)

- Hope to have relations to Donaldson-Thomas type
  invariants for (Calabi-Yau) 3-fold.
  possibly relevant to refined version

<u>Remark</u>. blow-up of a Calabi-Yau surface is not CY.

Warm-up (rank 1 case : Hilbert scheme of points)

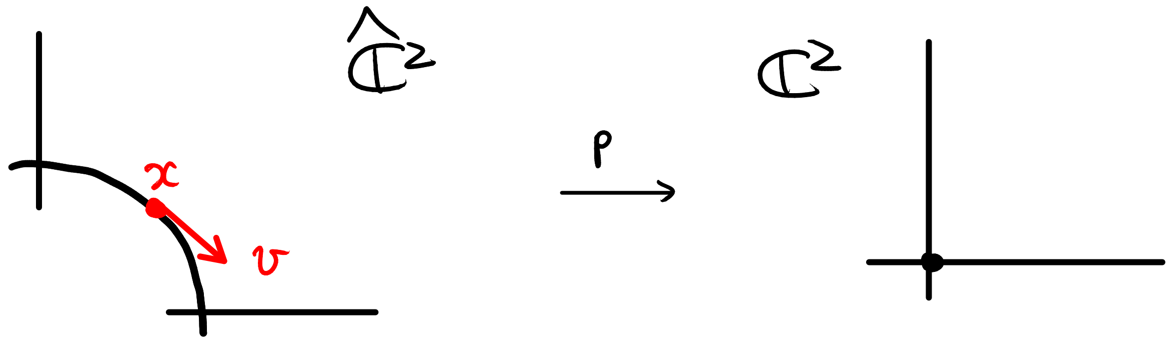
$X^{[n]}$: Hilbert scheme of n points in X
$= \{ \mathcal{I} \subset \mathcal{O}_X$: ideal sheaf, colength $= n \}$
...... crepant resolution of $S^n X = X^n / S_n$

e.g. $X^{[2]}$: either 2 distinct points $\{x, y\}$
or projectified tangent vector $\mathbb{C}v \subset T_x X$

We have a morphism $S^n \hat{X} \longrightarrow S^n X$.

But we do not have $\hat{X}^{[n]} \longrightarrow X^{[n]}$.

e.g. $n=2$



We cannot map $(\mathbb{C}v \subset T_x \hat{X}) \in \hat{X}^{[2]}$ to $X^{[2]}$
as $dp(v) = 0$.

Therefore $\hat{X}^{[n]} \dashrightarrow X^{[n]}$ is only defined partly.
(birational map)

$p_*(\mathcal{I})$ does not behave well in families,
as $R^1 p_*(\mathcal{I})$ may not vanish.

This makes the relation nontrivial and interesting.

For example, we know interesting relations among invariants:

<u>FAMOUS Göttsche formula</u>

$$\sum_{n=0}^{\infty} e(X^{[n]})\, q^n = \prod_{d=1}^{\infty} (1-q^d)^{-e(X)}$$

We have $e(\hat{X}) = e(X) + 1$.

$$\therefore \quad \frac{\sum_{n=0}^{\infty} e(\hat{X}^{[n]})\, q^n}{\sum_{n=0}^{\infty} e(X^{[n]})\, q^n} = \prod_{d=1}^{\infty} \frac{1}{1-q^d}$$

(essentially Dedekind η-function)

§ Perverse coherent sheaves on blow-up

As I told you, there is no morphism $\hat{X}^{[n]} \to X^{[n]}$.

Idea: Use perverse coherent sheaves on blow-up and wall-crossing to analyse relations.

Recall
$$\mathcal{C} := \{ K \in \mathrm{Coh}\, Y \mid \mathbb{R}p_* K = 0 \}$$
$$\mathcal{T} := \{ E \in \mathrm{Coh}\, Y \mid R^1 p_* E = 0,\ \mathrm{Hom}(E, K) = 0 \ \forall K \in \mathcal{C} \}$$
$$\mathcal{F} := \{ E \in \mathrm{Coh}\, Y \mid p_* E = 0 \}$$
$$\mathrm{Perv}(Y/X) = \left\{ E \in D^b(Y) \;\middle|\; \begin{array}{l} H^i(E) = 0 \quad i \neq 0, -1 \\ H^{-1}(E) \in \mathcal{F}, \quad H^0(E) \in \mathcal{T} \end{array} \right\}$$

When $Y = \hat{X}$

<u>Lemma</u>. $\mathcal{C} = \{ \mathcal{O}_C(-1)^{\oplus s} \mid s \in \mathbb{Z}_{\geq 0} \}$

<u>Cor</u>. $$\mathrm{Perv}(\hat{X}/X) = \left\{ E \in D^b(\hat{X}) \;\middle|\; \begin{array}{l} H^i(E) = 0 \quad i \neq 0, -1 \\ p_*(H^{-1}(E)) = 0, \quad R^1 p_*(H^0(E)) = 0 \\ \mathrm{Hom}(H^0(E), \mathcal{O}_C(-1)) = 0 \end{array} \right\}$$

In fact, this is automatic.

Let $c \in H^*(\hat{X})$.

$\hat{M}^0(c)$ = moduli space of stable perverse coherent sheaves E on $\hat{X}$ with $ch(E) = c$

more generally $\hat{M}^m(c) := \hat{M}^0(c \cdot e^{-m[C]})$ i.e. $E(-mC)$: stable perv. coh.

[<u>Rem</u>. category of perverse coherent sheaves is <u>not</u> invariant under $\otimes\, \mathcal{O}(-C)$

We have $\hat{M}^0(c) \longrightarrow M_H^X(p_*c)$.

Main Observation (Assume $GCD(r, (c_1, H)) = 1$)

1) $\exists$ diagram

$$\begin{tikzcd}[column sep=small] & \hat{M}^{m}(c) \arrow[dl] \arrow[dr] & & \hat{M}^{m+1}(c) \arrow[dl] \arrow[dr] & & \hat{M}^{m+2}(c) \arrow[dl] \arrow[dr] & \\ \cdots & & \hat{M}^{m,m+1}(c) & & \hat{M}^{m+1,m+2}(c) & & \cdots \end{tikzcd}$$

induced by the wall-crossing

2) $m \gg 0$ (compared with c)

w.r.t. H_ε

$\hat{M}^m(c) =$ moduli space of stable torsion-free sheaves on $\hat{X}$

3) If $(c_1, [C]) = 0$,

$\hat{M}^0(c) \cong M^X_H(p_* c)$

moduli space of stable torsion-free sheaves on X

Suppose $E^- \in \hat{M}^m(c) \setminus \hat{M}^{m+1}(c)$.

$\Longrightarrow \exists$ short exact sequence

$$0 \to \mathcal{O}_C(-m-1)^{\oplus n} \to E^- \to E' \to 0$$

$$E' \in \hat{M}^m(c) \cap \hat{M}^{m+1}(c)$$

For $E^+ \in \hat{M}^{m+1}(c) \setminus \hat{M}^m(c)$

$\Longrightarrow \exists$ short exact sequence

$$0 \to E' \to E^+ \to \mathcal{O}_C(-m-1)^{\oplus n} \to 0$$

$$E' \in \hat{M}^m(c) \cap \hat{M}^{m+1}(c)$$

$\hat{M}^{m,m+1}(c)$: parametrises $\{E' \oplus \mathcal{O}_C(-m-1)^{\oplus n}\}$ (n may vary)

$$E' \in \hat{M}^m(c) \cap \hat{M}^{m+1}(c)$$

This is the easiest case of the wall-crossing formula since $\mathcal{O}_C(-m-1)$ does not have the self-extension.

<u>Wall-crossing formula</u>

- Betti numbers (virtual Hodge polynomials)

For simplicity rank 1 case i.e. $c = 1 - N\, \mathrm{P.D.}[pt]$ $(N \in \mathbb{Z}_{\geq 0})$

$$\boxed{\frac{\sum_{N=0}^{\infty} P_t(\hat{M}^m(1-N[pt]))\, q^N}{\sum_{N=0}^{\infty} P_t(\hat{M}^{m-1}(1-N[pt]))\, q^N} = \frac{1}{1-t^{2m}q^m}}$$

<u>Rem</u> $m \to \infty$ compatible with the famous Göttsche's formula

$$\frac{\sum P_t(\hat{X}^{[n]})\, q^n}{\sum P_t(X^{[n]})\, q^n} = \prod_{m=1}^{\infty} \frac{1}{1-t^{2m}q^m}$$

<u>Rem</u>. I am not sure whether this follows from KS formula.
But anyway very easy to prove.
enough to show X = toric surface
use torus action

<u>Rem</u>. Göttsche's formula can be understood as a representation of Heisenberg algebra (oscillator). How about wall-crossing?

Blow-up formula for Donaldson type invariants

comparing $\int_{\hat{M}^m(c)} \Phi(\mathcal{E})$ and $\int_{\hat{M}^{m+1}(c)} \Phi(\mathcal{E})$.

$\mathcal{E}$: universal family over $\hat{M}^m(c)$

$\Phi(\mathcal{E})$: a cohomology class constructed from $\mathcal{E}$ e.g. $ch(\mathcal{E})/\Sigma$

$\Sigma \in H_*(\hat{X})$

The integral are not motivic at all.

The approach is based on Mochizuki's master spaces.

(We need to use the ADHM type description by a technical reason.)

$$\int_{\hat{M}^{m+1}(c)} \Phi(\mathcal{E}) - \int_{\hat{M}^m(c)} \Phi(\mathcal{E})$$

$$= \sum_{j=1}^{\infty} \frac{1}{j!} \int_{\hat{M}^m(c - j\,ch(\mathcal{O}_C(-m-1)))} \operatorname*{Res}_{\hbar_j=0} \cdots \operatorname*{Res}_{\hbar_1=0} \left[\Phi\left(\mathcal{E}_b \oplus \bigoplus_{i=1}^{j} \mathcal{O}_C(-m-1) \boxtimes e^{-\hbar_i}\right) \times \frac{\prod_{1 \le i_1 \ne i_2 \le j} (-\hbar_{i_1} + \hbar_{i_2})}{\prod_{i=1}^{j} e(\mathrm{Ext}^1(\mathcal{E}_b, \mathcal{O}_C(-m-1) \boxtimes e^{-\hbar_i}))\, e(\mathrm{Ext}^1(\mathcal{O}_C(-m-1) \boxtimes e^{-\hbar_i}, \mathcal{E}_b))} \right]$$

$\hat{M}^m(c - j\,ch(\mathcal{O}_C(-m-1))) \times \{ \bigoplus_{i=1}^{j} \mathcal{O}_C(-m-1) \boxtimes e^{-\hbar_i} \}$: "piece" of $\hat{M}^{m,m+1}(c)$

$\mathcal{E}_b$: universal family for $\hat{M}^m(c - j\,ch(\mathcal{O}_C(-m-1)))$

"normal bundle"

This gives us a recursive formula expressing

$$\int_{\hat{M}^{\hat{\times}}_{H_\varepsilon}(c)} \Phi(\mathcal{E}) \quad \text{in terms of} \quad \int_{M^{\times}_{H}(c')} \Phi'(\mathcal{E}) \quad \text{for various } \Phi',\ c'.$$

But they are very complicated.

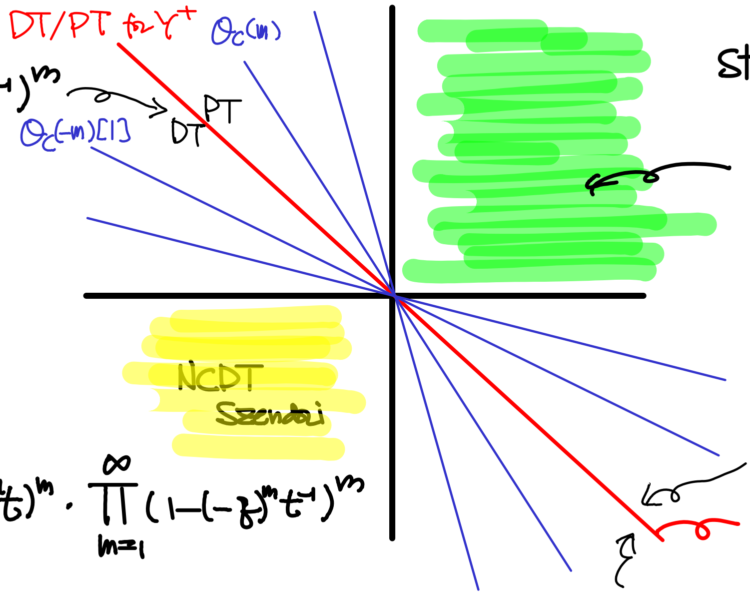
We can recover FS blow-up formula or its variations, but we need to combine above with something completely different.

Remark $X =$ resolved conifold $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$

We have a similar wall-crossing formula for DT type invariants [Nagao – N]

$$\mathrm{Stab}(X)_{\mathbb{R}} = \mathbb{R}^2$$

$$Z = M(-q)^2 \prod_{m=1}^{\infty} (1-(-q)^m t^{-1})^m$$



$$Z = 1$$

$$Z = \prod_{m=1}^{\infty} (1-(-q)^m t)^m$$

$$Z = M(-q)^2 \cdot \prod_{m=1}^{\infty} (1-(-q)^m t)^m \cdot \prod_{m=1}^{\infty} (1-(-q)^m t^{-1})^m$$

$$Z = M(-q)^2 \cdot \prod_{m=1}^{\infty} (1-(-q)^m t)^m$$